Quantum CPOs

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Abstract

This submission is an extended abstract of [2], and introduces the monoidal closed category **qCPO** of *quantum cpos*, whose objects are 'quantized' analogs of ω-complete partial orders (cpos). The category **qCPO** is enriched over the category **CPO** of cpos, and contains both **CPO**, and the opposite of the category **FdAlg** of finite-dimensional von Neumann algebras, as monoidal subcategories. The category **qCPO** enjoys the same properties that make **CPO** so useful for the semantics of higher-order programming languages that support recursion. Since every finite-dimensional von Neumann algebra is a quantum cpo, **qCPO** is a natural candidate for modeling higher-order quantum programming languages that support recursion. Indeed, we use **qCPO** to construct a sound model for the quantum programming language Proto-Quipper-M (PQM) extended with term recursion. Likewise, **qCPO** admits a sound and computationally adequate model for LNL-FPC, which is both an extension of FPC with linear types, and an extension of a circuit-free fragment of PQM that includes recursive types.

1 Introduction

Proto-Quipper-M (PQM) and its relatives [5] are typical quantum programming languages, whose models consist of a linear/non-linear adjunction, i.e., a monoidal adjunction

$$\mathbf{C} \rightleftarrows \mathbf{L} \tag{1}$$

between a Cartesian closed category $\mathbf{C}$ and a symmetric monoidal closed category $\mathbf{L}$, where the latter contains a suitable monoidal subcategory representing quantum circuits, for instance $\mathbf{M} := \mathbf{FdAlg}^{\mathrm{op}}$, the opposite of the category of finite-dimensional operator algebras and unital †-homomorphisms.

In [3], PQM was extended with term recursion, and in [4] the circuit-free fragment of PQM was extended with type recursion. Furthermore, the latter article also presented conditions on the category $\mathbf{L}$ for which the model (1) is sound and computationally adequate, provided that one chooses $\mathbf{C}$ to be the category **CPO** of ω-complete partial orders (cpos) and Scott continuous maps. Problematically, none of the standard choices for $\mathbf{L}$ containing $\mathbf{M}$ as a monoidal subcategory (such as the category of von Neumann algebras or the category of presheaves on $\mathbf{M}$) satisfies these conditions.

We propose a new monoidal closed category **qCPO** of quantum cpos and Scott continuous maps to address this issue. This category contains $\mathbf{M}$ as a monoidal subcategory, and can be regarded as a 'quantum generalization' of **CPO**; hence, there is an inclusion functor of **CPO** into **qCPO**. This inclusion is strong monoidal, and has a right adjoint, and these two functors together form a linear/non-linear adjunction. Moreover, just as **CPO** has a subcategory $\mathbf{CPO}_{\perp!}$ of pointed cpos with strict maps, **qCPO** has a subcategory $\mathbf{qCPO}_{\perp!}$ of pointed quantum cpos and strict maps, which contains $\mathbf{CPO}_{\perp!}$ as a monoidal subcategory. In fact, there exists a 'lifting' functor $(-)_\perp : \mathbf{qCPO} \to \mathbf{qCPO}_{\perp!}$ that restricts to the ordinary lifting functor $(-)_\perp : \mathbf{CPO} \to \mathbf{CPO}_{\perp!}$. Combining this lift with the inclusion $\mathbf{CPO} \to \mathbf{qCPO}$ gives us a linear/non-linear model

$$\mathbf{CPO} \rightleftarrows \mathbf{qCPO}_{\perp!}, \tag{2}$$

that satisfies all of the conditions in [4]. As a direct application of the results from [3] and [4], we obtain:

Theorem 1.1 *The linear/nonlinear adjunction (2) is both a sound model for PQM with term recursion, and a model for LNL-FPC that is computationally adequate at non-linear types.*

We are not aware of any other models satisfying all the conclusions of the above theorem. We expect to be able to prove even stronger statements, namely that (2) is a computationally adequate model at all types for PQM extended with recursive types. We also expect to model state preparation.

2 Quantum Sets

In order to describe our quantization of the concept 'cpo', we first need to discuss the quantization of the concepts 'set', 'relation' and 'function'. We now recall the quantum generalizations of these concepts given in [1].

A *quantum set* is completely determined by a set $\mathrm{At}(\mathcal{X})$ of nonzero finite-dimensional Hilbert spaces, called the *atoms* of $\mathcal{X}$. Even though on a formal, set-theoretic level, every quantum set $\mathcal{X}$ is simply a set of Hilbert spaces, we prefer to draw a distinction between $\mathrm{At}(\mathcal{X})$ and $\mathcal{X}$, with the former an object of the familiar category **Set**, and the latter an object of its quantum generalization **qSet**, to be defined.

Given two finite-dimensional Hilbert spaces X and Y, we notate the space of linear operators $X \to Y$ by $L(X,Y)$. We now define a *binary relation* R between quantum sets $\mathcal{X}$ and $\mathcal{Y}$ to be a function which assigns to each atom X of $\mathcal{X}$ and each atom Y of $\mathcal{Y}$ a subspace $R(X,Y)$ of $L(X,Y)$. These are essentially the quantum relations of Weaver [6]. Quantum sets and binary relations form a category **qRel**, if we define the composition the $S \circ R$ of two relations, R from $\mathcal{X}$ to $\mathcal{Y}$, and S from $\mathcal{Y}$ to $\mathcal{Z}$, by $(S\circ R)(X,Z) = \mathrm{span}\{sr \mid \exists Y \in \mathrm{At}(\mathcal{Y}).\, r \in R(X,Y)\,\&\, s \in S(Y,Z)\}$. The identity binary relation $I_{\mathcal{X}}$ on $\mathcal{X}$ is defined by $I_{\mathcal{X}}(X,X) = \mathbb{C}1_X$, where $1_X : X \to X$ is the identity operator, with $I_{\mathcal{X}}(X,X') = 0$ whenever $X \neq X'$. The ordering $\leq$ on the subspaces of Hilbert spaces induces an ordering on the homset $\mathbf{qRel}(\mathcal{X},\mathcal{Y})$, which we notate by $\leq$ as well, and which makes the homset a complete lattice. We write $\bigvee$ and $\bigwedge$ for the supremum and infimum respectively of this ordering.

Similar to **Rel**, the category **qRel** is a dagger-compact category. We can embed **Rel** into **qRel** as follows. To each ordinary set S we assign the quantum set $`S$, defined by $\mathrm{At}(`S) = \{\mathbb{C}_s : s \in S\}$, where $\mathbb{C}_s$ denotes a one-dimensional Hilbert space labeled by the element s. Furthermore, given a binary relation r between ordinary sets R and S, we define the quantum relation $`r$ between $`S$ and $`T$ by $(`r)(\mathbb{C}_s,\mathbb{C}_t) = L(\mathbb{C}_s,\mathbb{C}_t)$ if $(s,t) \in r$ and $(`r)(\mathbb{C}_s,\mathbb{C}_t) = 0$ otherwise. In this way, we obtain a fully faithful functor $`(-) : \mathbf{Rel} \to \mathbf{qRel}$ that preserves the dagger monoidal structure.

An ordinary function between ordinary sets is formally defined to be a binary relation of a particular kind, and we use essentially the same characterization for functions between quantum sets. Hence we define a *function* $F : \mathcal{X} \to \mathcal{Y}$ between quantum sets $\mathcal{X}$ and $\mathcal{Y}$ to be a binary relation $F \in \mathbf{qRel}(\mathcal{X},\mathcal{Y})$ such that

$$F^\dagger \circ F \geq I_{\mathcal{X}}, \quad \text{and} \quad F \circ F^\dagger \leq I_{\mathcal{Y}}.$$

The composition of two such functions is another such function, and we notate the category of quantum sets and functions by **qSet**.

The category **qSet** is complete and cocomplete, and symmetric monoidal closed. Moreover, the functor $`(-) : \mathbf{Rel} \to \mathbf{qRel}$ restricts to a strong monoidal functor $\mathbf{Set} \to \mathbf{qSet}$ that has a right adjoint. Thus, the resulting adjunction forms an LNL-model. Moreover, to each quantum set $\mathcal{X}$, we can assign a von Neumann algebra $\ell^\infty(\mathcal{X}) = \bigoplus_{X \propto \mathcal{X}} L(X)$, the ℓ^∞-direct sum of von Neumann algebras, and this assignment extends to a fully faithful functor $\ell^\infty : \mathbf{qSet}^{\mathrm{op}} \to \mathbf{WStar}_{\mathrm{NMIU}}$. We obtain a duality if we corestrict this functor to the full subcategory of *hereditarily atomic* von Neumann algebras, i.e., those von Neumann algebras for which any von Neumann subalgebra is atomic. Up to isomorphism, such von Neumann algebras are exactly the ℓ^∞-direct sums of matrix algebras.

3 Quantum CPOs

Following Weaver [6, Definition 2.6], we define a *quantum poset* to be a pair $(\mathcal{X}, R)$ consisting of a quantum set $\mathcal{X}$ and a relation $R \in \mathbf{qRel}(\mathcal{X}, \mathcal{X})$ such that

$$I_{\mathcal{X}} \leq R, \qquad R \circ R \leq R, \qquad \text{and} \qquad R \wedge R^{\dagger} \leq I_{\mathcal{X}},$$

formalizing reflexivity, transitivity, and antisymmetry, respectively. A *monotone* map $F : (\mathcal{X}, R) \to (\mathcal{Y}, S)$ is simply a function $F : \mathcal{X} \to \mathcal{Y}$ satisfying $F \circ R \leq S \circ F$. We notate the category of quantum posets with monotone maps by **qPOS**. Moreover, we have a fully faithful functor $`(-) : \mathbf{POS} \to \mathbf{qPOS}$ defined by $(S, \sqsubseteq) \mapsto (`S, `\sqsubseteq)$.

Theorem 3.1 *The category* **qPOS** *is complete, has all coproducts, and is monoidal closed.*

Let $(\mathcal{X}, R)$ be a quantum poset, and let $\mathcal{W}$ be any quantum set. The quantum analog of an increasing sequence in a CPO is an increasing sequence of test functions: Let $K_1 \sqsubseteq K_2 \sqsubseteq \cdots : \mathcal{W} \to \mathcal{X}$ be a monotonically increasing sequence, where the order on $\mathbf{qSet}(\mathcal{W}, \mathcal{X})$ is defined by $F \sqsubseteq G$ iff $G \leq R \circ F$.[1] If there exists a function $K_\infty : \mathcal{W} \to \mathcal{X}$ such that $R \circ K_\infty = \bigwedge_{n \in \mathbb{N}} R \circ K_n$, then we say that K_∞ is the *limit* of the sequence, and we write $K_n \nearrow K_\infty$. If for each quantum set $\mathcal{W}$, every monotonically increasing sequence $\mathcal{W} \to \mathcal{X}$ has a limit, then we say that $\mathcal{X}$ is a *quantum cpo*. Furthermore, a function $F : (\mathcal{X}, R) \to (\mathcal{Y}, S)$ between quantum cpos is said to be *Scott continuous* if for each quantum set $\mathcal{W}$ and each monotonically increasing sequence $K_1 \sqsubseteq K_2 \sqsubseteq \cdots \sqsubseteq K_\infty : \mathcal{W} \to \mathcal{X}$ the condition $K_n \nearrow K_\infty$ implies the condition $F \circ K_n \nearrow F \circ K_\infty$. We write **qCPO** for the category of quantum cpos and Scott continuous functions.

Theorem 3.2 **qCPO** *is monoidal closed, complete, and has all coproducts. Moreover, the functor* $`(-) :$ $\mathbf{POS} \to \mathbf{qPOS}$ *restricts to a strong monoidal fully faithful functor* $\mathbf{CPO} \to \mathbf{qCPO}$ *that has a right adjoint.*

Any computationally adequate model of a programming language supporting recursion needs denotations for nonterminating terms. In ordinary domain theory, one passes to the subcategory $\mathbf{CPO}_{\perp!}$ of pointed cpos with strict Scott continuous maps in order to have these denotations. Similarly, we can define a subcategory $\mathbf{qCPO}_{\perp!}$ of pointed quantum cpos and strict Scott continuous maps. Moreover, $`(-)$ restricts to a fully faithful strong monoidal functor $\mathbf{CPO}_{\perp!} \to \mathbf{qCPO}_{\perp!}$ that has a right adjoint, and we have a lifting functor $(-)_\perp : \mathbf{qCPO} \to \mathbf{qCPO}_{\perp!}$ generalizing the ordinary lifting $\mathbf{CPO} \to \mathbf{CPO}_{\perp!}$ in the sense that $(\mathbf{CPO} \xrightarrow{`(-)} \mathbf{qCPO} \xrightarrow{(-)_\perp} \mathbf{qCPO}_{\perp!}) = (\mathbf{CPO} \xrightarrow{(-)_\perp} \mathbf{CPO}_{\perp!} \xrightarrow{`(-)} \mathbf{qCPO}_{\perp!})$. Finally, the resulting functor $\mathbf{CPO} \to \mathbf{qCPO}_{\perp!}$ forms the left adjoint part of the model (2).

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[1] This is the quantum rendering of the ordinary case, where $f \leq g$ pointwise iff $g(x) \in \uparrow f(x)$ for each x.